Taipei Exchange - Zero Coupon Yield Curve

■ Source of pricing data for ZCYC construction

The source is daily pricing data for the government bond index on Taipei Exchange (TPEx). The sample is all bonds in the government bond index.

■ Model for ZCYC construction

Steeley B-spline (basis spline) model

Spline functions are discussed and studied in more areas than others. They are also the most versatile. The term "spline" comes from construction, where it refers to the practice of using a thin strip of wood bent in several fixed points to draw a smooth curve. The curve can take many shapes by changing the number of fixed points and their locations. The spline-based yield curve fitting method follows the Weierstrass approximation theorem. By the theorem, every continuous function can be uniformly approximated as closely as desired by a polynomial function. It states that any continuous function f defined on a closed interval $[a,b]$ can be uniformly approximated by a polynomial function P_n.

$$E_n(f) = \max_{a \le x \le b} \left| f(x) - P_n(x) \right| = \left\| f - P \right\|_{\infty}$$

When $n \to \infty$, the error $\to 0$.

In general, higher degree polynomials fit the data better. However, if the degree of freedom gets too large, the high degree polynomial often sees the two tails become much more uncertain. To reduce uncertainty in the tails, a piecewise polynomial is used for approximation. It effectively reduces the degree of the piecewise polynomial and uncertainty in the tails while still achieving a good fit. Therefore, the spline approach is selected to fit the yield curve. The time period is usually divided into several intervals. The points between intervals are known as knot points (ξ_i). The intervals are then fitted individually.

In 1991, Steeley used the U.K. gilt-edged bond to fit the gilt-edged term structure. The selection of a cubic function as the basis function followed the definition of B-spline by Powell in 1981. The B-spline function is as follows:

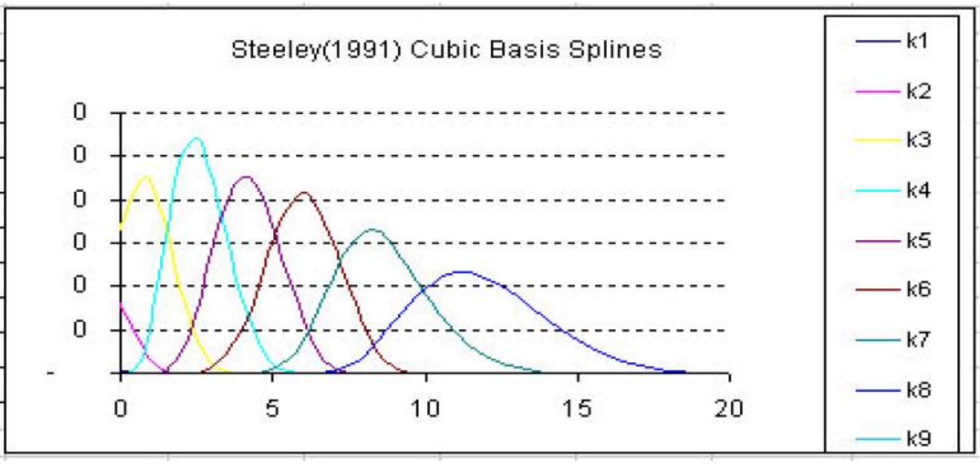
On the closed interval $[A,B]$, given node ξ_i (i = 01,....,n) and if $A = \xi_0 < \xi_1 < ..... < \xi_n = B$ is true, a kth degree B-spline function $B_i^k(t)$ is defined as:

$$B_i^k(t) = \sum_{j=0}^{k+1} \left\{ \left[\prod_{l=0, l \ne j}^{k+1} \frac{1}{\xi_{i+l} - \xi_{i+j}} \right] \left[\max\left(t - \xi_{i+j}, 0\right) \right]^k \right\}$$

In general, the B-spline above is known as a linear B-spline if k=l, quadratic B-spline if k=2, and cubic B-spline if k=3. Furthermore, the B-spline $B_i^k(t)$ is not zero only when time (t) is in (ξ_i, ξ_{i+k+1}). $B_i^k(t)$ is zero otherwise. This is known as local support.

Steeley set the B-spline to be a cubic function. In other words, the basis function for a cubic spline is as shown below.

Knot (j)	
κ_1	-
κ_2	1.0
κ_3	2.0
κ_4	4.0
κ_5	6.0
κ_6	8.0
κ_7	10.0
κ_8	15.0
κ_9	20.0



Being locally supported, B-spline offers better computational stability than general spline functions. It produces more accurate approximation given the degree (k) of the function remains the same.

B-spline requires n+k linearly independent kth-degree basis functions to approximate the curve to be fitted. However, only n-k linearly independent basis functions exist with nodes ξ_i (i= 0,1,....,n) on the closed interval [A,B]. It is therefore necessary to create k new nodes beyond ξ_0 and ξ_n at the ends. If k=3, nodes ξ_i (i = -3,-2,-1,n+1,n+2,n+3) have to be added to create 2k linearly independent basis functions. They, together with the n-k linearly independent basis function on the closed interval [A,B], form n+k linearly independent k-degree basis functions.

To use the spline model to fit the yield curve, the underlying discount function is as follows:

$$D(t) = 1 + \sum_{i=1}^{n} a_i f_i(t)$$

with a_i i=1~n as the parameter

$f_i(t)$ is a specific function, which is the basis function above.

The optimal solution is where the following is true for a_i i=1~n.

$$Min\left(\sum_{i=1}^{n} w_i \cdot \left(\hat{P}_i - P_i\right)^2\right)$$

where $$w_i = \frac{1/D_j}{\sum_{j=1}^{n} 1/D_j}$$ D_j is the duration (MaCauley Duration) of the j_{th} series of government bond.

The discount function of the best fit can be derived from the optimal solution, and then entered into the equation below to get zero coupon rates.

$$R_t = D(t)^{-\frac{1}{t}} - 1$$

R_t is the zero coupon rate at t

Svensson model

In 1994, the Svensson model added another hump or U-shape to the Nelson-Siegel model. Therefore, yield curve fitting using the Svensson model produces two turning points. It allows more flexibility and variety in yield curves.

The Svensson model extends the instantaneous forward rate into the equation below:

$$f(TTM) = \beta_0 + \beta_1 \cdot \left(e^{-\left(\frac{TTM}{\tau_1}\right)} \right) + \beta_2 \cdot \left[\frac{TTM}{\tau_1} \left(e^{-\left(\frac{TTM}{\tau_1}\right)} \right) \right] + \beta_3 \cdot \left[\frac{TTM}{\tau_2} \left(e^{-\left(\frac{TTM}{\tau_2}\right)} \right) \right]$$

The parameters to estimate the instantaneous forward rate are β_0, β_1, β_2, β_3, τ_1, and τ_2, which are defined as follows:

β_0: This parameter must be positive. It is the asymptotic value of $f(TTM)$, which means the instantaneous forward rate should converge to β_0 in the long run.

β_1: This parameter is a short term factor. It determines the difference between the short term (or initial) value and the asymptotic value. It also determines the speed of convergence of the short term value to the asymptotic value. If β_1 is negative, $f(TTM)$ has a positive slope, and vice versa.

τ_1: This parameter must be positive. It determines the position of the first hump or U-shape on the curve and the speed of convergence for β_1. A small τ_1 indicates fast convergence, and vice versa.

β_2: This parameter describes the size and direction of the kurtosis of the first turning point on the curve. The curve shows a hump if it is positive, or a U-shape if it is negative.

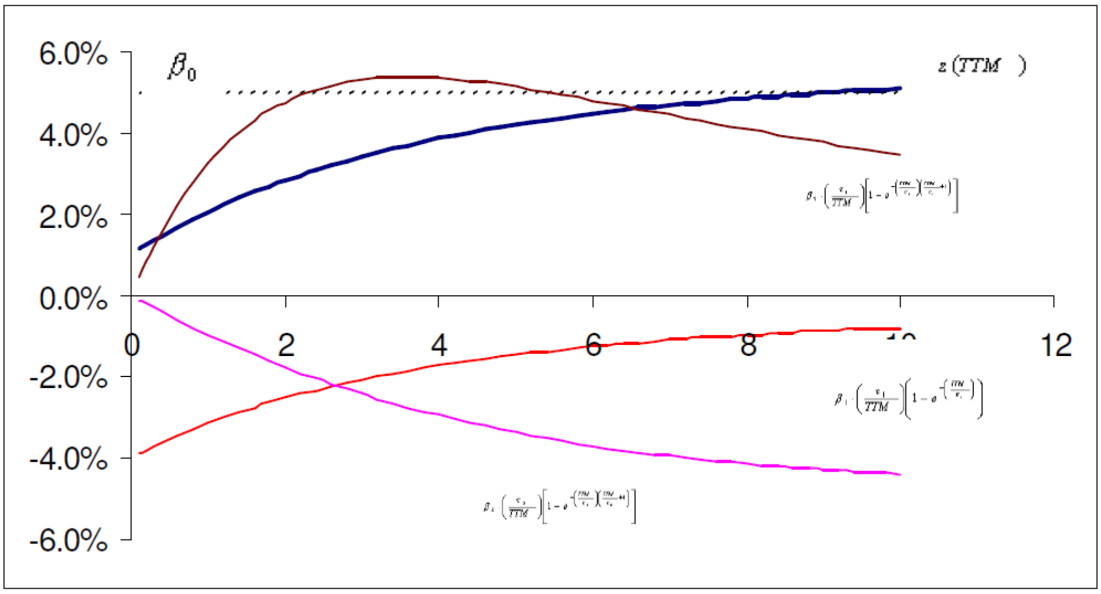
τ_2: This parameter must be positive. It determines the position of the second hump or U-shape on the curve.

β_3: This parameter describes the size and direction of the kurtosis of the second turning point on the curve. The curve shows a hump if it is positive, or a U-shape if it is negative.

The replacement of the instantaneous forward rate $f(TTM)$ by spot rate $z(TTM)$ can be expressed as follows:

$$z(TTM) = \beta_0 + \beta_1 \cdot \left(\frac{\tau_1}{TTM} \right) \left(1 - e^{-\left(\frac{TTM}{\tau_1}\right)} \right) + \beta_2 \cdot \left(\frac{\tau_1}{TTM} \right) \left[1 - e^{-\left(\frac{TTM}{\tau_1}\right)} \left(\frac{TTM}{\tau_1} + 1 \right) \right] + \beta_3 \cdot \left(\frac{\tau_2}{TTM} \right) \left[1 - e^{-\left(\frac{TTM}{\tau_2}\right)} \left(\frac{TTM}{\tau_2} + 1 \right) \right]$$

The zero coupon yield rate is constructed as follows:



β_0	5.0%
β_1	-4.0%
β_2	18.0%
β_3	-15.0%
τ_1	2.000
τ_2	7.000

In fact, the last item in the instantaneous forward rate *f*(*TTM*) in the Svensson model was added to provide a better fit and not economically meaningful. Hence, the Svensson model is not exactly an extension of the Nelson-Siegel model. However, the greater variety of yield curves available in the Svensson model makes it a better fitting model than the Nelson-Siegel model.